Math 249 Lecture 33 Notes

Daniel Raban

November 8, 2017

1 Chamber Walk Length and Iwahori's Theorem

1.1 Length of chamber walks

Last time we talked about the chambers formed when you chop up a space with finitely many hyperplanes intersecting at the origin. We saw:

- Chambers are simplicial.
- G acts transitively on the chambers.
- Any chamber walk can be factored as a product of simple reflections.
- If w is a chamber walk, $\ell(w) \leq$ number of hperplanes separating C from wC.

Definition 1.1. Let S be the set of simple reflections. Then (G, S) is called a *Coxeter system*.

Proposition 1.1. *If a chamber walk $w = s_{i_1} \cdots s_{i_\ell}$ crosses the same hyperplane H twice, then it is non-minimal.*

Proof. The idea is that if we cross H twice, we can take the path we took on the other side of H and reflect it over H, back onto the original side of H. This path should be the same length, except we don't waste two reflections crossing H. We need to check rigorously that the lengths are the same.

Let $s_{i_1} \cdots s_{i_{j-1}} C$ to $s_{i_1} \cdots s_{i_j} C$ be when we cross H, and let $s_{i_1} \cdots s_{i_k} C$ to $s_{i_1} \cdots s_{i_{k+1}} C$ be when we cross it again. The reflection $s_H = (s_{i_1} \cdots s_{i_{j-1}})_{s_{i_j}}$, where g_h means ghg^{-1} (conjugation). Similarly, $s_H = (s_{i_j} \cdots s_{i_{k-1}})_{s_{i_k}}$. So

$$s_{i_j} = s_{i_j} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \cdots s_{i_j}$$

$$1 = (s_{i_{j+1}} \cdots s_{i_{k-1}})(s_{i_k} s_{i_{k-1}} \cdots s_{i_j})$$

$$s_{i_j} \cdots s_{i_k} = s_{i_{j+1}} \cdots s_{i_{k-1}}.$$

□

Corollary 1.1. *$\ell(w)$ equals the number of hyperplanes separating C from wC.*

Corollary 1.2. *If $wC = C$, then $w = 1$.*

Example 1.1. Let $G = S_n$. The simple reflections are $s_i = (i\ \ (i+1))$. $\ell(w)$ is the number of inversions of w.

1.2 Iwahori's theorem

If s_i, s_j generate a copy of D_{2m}, then $s_i s_j$ has order m. Writing this out, we have $s_i s_j s_i s_j \cdots s_i s_j = 1$. If we cut this in half and multiply by the inverse of the right, we get $s_i s_j s_i \cdots = s_j s_i \cdots$, where both these have m terms. We also have that $s_i^2 = 1$.

Definition 1.2. A *Coxeter move* is an element $s_i s_j s_i \cdots = s_j s_i \cdots$.

Theorem 1.1 (Iwahori)**.** *Let G be a Coxeter group.*

1. *G has a presentation $\langle s \mid s_i^2 = 1, s_i s_j s_i \cdots = s_j s_i \cdots \rangle$.*
2. *Every two minimal factorizations of $w \in G$ are connected by a sequence of Coxeter moves.*

Proof. We proceed by induction on $\ell(w)$. Consider two factorizations $w = s_i U = s_j V$ that start with different reflections s_i, s_j. Then consider the chamber walks they generate. This means that $s_i C$ and $s_j C$ both separate C from wC, so w_C is in the chamber C' with walls corresponding to s_i and s_j. Doing the reflections $s_i s_j \cdots$ (or $s_j s_i \cdots$) places us in C', so we can start the walk with either of those two moves. This gives us

$$w = s_i s_j \cdots X = s_j s_i \cdots X.$$

So $U = s_j \cdots X$ and $V = s_i \cdots X$, and the inductive hypothesis takes care of the rest. $\square$

Example 1.2. Let $G = S_n$. This gives us that $s_i^2 = 1$, $s_i s_j = s_j s_i$ if $|i - j| > 1$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Example 1.3. Let $G = B_n$ with the reflections $s_1, \ldots, s_{n-1}$ and s_n, which sends $n \mapsto \bar{n}$. Then $s_i s_n = s_n s_i$ for $i < n - 1$, and $s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}$.